Estimating Gaussian noise standard deviation from sample and order statistics

Cheng Guan Koay*

June 16, 2009

The main objective of writing this note is to share with interested readers some of the simpler techniques for estimating the Gaussian noise SD together with all the relevant numerical constants and scaling factors. Here, we are going to touch on *simple* techniques for estimating Gaussian noise standard deviation (SD) from a collection of noise-only measurements obtained from a magnitude-reconstructed image or multiples of these images. By *simple*, I mean techniques that are based upon sample or order statistics. I am aware of the following techniques, grouped according to the statistics used in the estimation:

- **Sample mean**. Estimating the Gaussian noise SD, σ_g, through the sample mean is quite well known. In the context MRI, the formula for the single channel MRI system dates back to the 80s, e.g., [1]. The formula for the multichannel phased array MRI system that based on the sum-of-squares algorithm of [2] was developed in the 90's, e.g., [3].

  Let the sample mean of a collection of noise-only measurements be denoted by $\langle M \rangle$. The Gaussian noise SD, σ_g, can be estimated in terms of $\langle M \rangle$ through the following equation:

  $$\sigma_g = \langle M \rangle / \beta_N, \tag{1}$$

  where N is the number of combined channels and β_N is given by

  $$\beta_N = \sqrt{\pi/2}\frac{(2N-1)!!}{2^{N-1}(N-1)!}, \tag{2}$$

  $$= \sqrt{\frac{\pi}{2}}\prod_{k=2}^{N}\frac{k-1/2}{k-1}. \tag{3}$$

  The notation here is similar to our previous works on this area of research, e.g., [4, 5]. Note that Eq.(3), which appeared in [5], is easier to compute than Eq.(2), particularly when N is large, say 64. For convenience, the derivation of Eq.(3) is presented in the Appendix. Selected values of β_N is shown in Table 1.

- **Sample standard deviation**. Estimating the Gaussian noise SD, σ_g, through the sample standard deviation is also quite well known, e.g, [1, 3].

*Feel free to contact me with your comments. Contact info: guankoac@mail.nih.gov.

Table 1: Numerical values of β_N

N	β_N
1	1.253314
2	1.879971
4	2.741625
8	3.938026
16	5.612839
32	7.968812
64	11.29163

Let the sample standard deviation of a collection of noise-only measurements be denoted by σ_M. The Gaussian noise SD,σ_g, can be estimated through σ_{M_N} by:

$$\sigma_g = \frac{\sigma_M}{\sqrt{\xi(0,N)}}, \tag{4}$$

where $\xi(\theta, N)$ is the correction (or scaling) factor first derived in our work,[4] and θ is the underlying signal-to-noise ratio. In the context of estimating the Gaussian noise SD, θ is assumed to be zero; hence, $\xi(0, N)$. Note that $\xi(0, N)$ can be simply given by $2N - \beta_N^2$. For convenience, selected values of the denominator of Eq.(4) is shown in Table 2.

Table 2: Numerical values of $\sqrt{\xi(0,N)}$

N	$\sqrt{\xi(0,N)}$
1	0.655136
2	0.682428
4	0.695337
8	0.7013944
16	0.704296
32	0.705714
64	0.706413

- **Sample median**.

  Estimating the Gaussian noise SD, σ_g, through the sample median is relatively recent, see [5]. One of the motivation is that median is more robust than sample mean or sample standard deviation against outliers or image artifacts, which are common in the background of many MR images. It turns out that there are other order statistics that is more optimal than the sample median, see [5], and we shall discuss this optimal order statistic in the next section.

  Let the sample median of a collection of noise-only measurements be denoted by μ. The Gaussian noise SD,σ_g, can be estimated in terms of μ through the following equation:

$$\sigma_g = \frac{\mu}{\sqrt{2P^{-1}(1/2|N,1)}}, \tag{5}$$

where $P_s^{-1}(1/2|N,1)$ is related to the inverse cumulative distribution function of the sum of squares of magnitude signals divided by $2\sigma_g^2 K$ and K is the number of terms in the summation. Interested readers are encouraged to refer to [5] for more in-depth discussion. Selected numerical values of the denominator of Eq.(5) is shown in Table 3.

Table 3: Numerical values of $\sqrt{2P^{-1}(1/2|N,1)}$

N	$\sqrt{2P^{-1}(1/2\|N,1)}$
1	1.177410
2	1.832128
4	2.710003
8	3.916439
16	5.597844
32	7.958302
64	11.284234

- **Sample quantile of optimal order**. As mentioned in the previous section, there are other order statistics that is more optimal than the sample medium. For a complete discussion, please refer to our recent work, [5]. It suffice to say that for each N the most optimal order statistics can be found. Let the quantile of optimal order, α^*, be denoted by q_{α^*}.

  The Gaussian noise SD,σ_g, can be estimated in terms of q_{α^*} through the following equation:

$$\sigma_g = \frac{q_{\alpha^*}}{\sqrt{2P^{-1}(\alpha^*|N,1)}}, \tag{6}$$

  where P_s^{-1} was defined previously. Selected numerical values of the denominator of Eq.(6) is shown in Table 4.

Table 4: Numerical values of $\sqrt{2P^{-1}(\alpha^*|N,1)}$

N	α^*	$\sqrt{2P^{-1}(\alpha^*\|N,1)}$
1	0.796812	1.785286
2	0.730630	2.275853
4	0.672195	3.028914
8	0.625403	4.143826
16	0.590048	5.759311
32	0.564177	8.072721
64	0.545561	11.365227

Finally, I will end this note by mentioning that there are more sophisticated means for estimating the Gaussian noise SD than the above mentioned techniques, e.g., [5, 6, 7].

Appendix

In this appendix, we will derive Eq.(3) from Eq.(2). Here are the steps:

$$\begin{aligned}
\beta_N &= \sqrt{\frac{\pi}{2}}\frac{(2N-1)!!}{2^{N-1}(N-1)!} && (7)\\
&= \sqrt{\frac{\pi}{2}}\frac{(2N-1)\times(2N-3)\times(2N-5)\times\cdots\times 3\times 1}{2^{N-1}(N-1)\times(N-2)\times(N-3)\times\cdots\times 2\times 1} && (8)\\
&= \sqrt{\frac{\pi}{2}}\frac{(2N-1)/2\times(2N-3)/2\times(2N-5)/2\times\cdots\times 3/2\times 1}{(N-1)\times(N-2)\times\cdots\times(2)\times(1)} && (9)\\
&= \sqrt{\frac{\pi}{2}}\frac{(2(N)-1)/2\times(2(N-1)-1)/2\times(2(N-2)-1)/2\cdots}{(N-1)\times(N-2)\times(N-3)\cdots\times(2)\times(1)} && (10)\\
&= \sqrt{\frac{\pi}{2}}\frac{((N)-1/2)\times((N-1)-1/2)\times((N-2)-1/2)\cdots}{((N)-1)\times((N-1)-1)\times((N-2)-1)\times\cdots} && (11)\\
&= \sqrt{\frac{\pi}{2}}\frac{(N-1/2)}{(N-1)}\times\frac{((N-1)-1/2)}{((N-1)-1)}\times\frac{((N-2)-1/2)}{((N-2)-1)}\times\ldots && (12)\\
&= \sqrt{\frac{\pi}{2}}\prod_{k=2}^{N}\frac{k-1/2}{k-1}. && (13)
\end{aligned}$$

Note that there are $N-1$ factors in $(2N-1)!!$, not counting the factor 1, and that every factor is an odd number for any positive integer N. By dividing each factor by 2 in a descending order starting from the largest factor, we arrive at Eq.(9) with 3/2 being the last factor after the division, again not counting the factor 1. This is the main reason why k in Eq.(13) starts at 2.

References

[1] W. A. Edelstein, P. A. Bottomley, L. M. Pfeifer, A signal-to-noise calibration procedure for nmr imaging systems, Medical Physics 11 (2) (1984) 180–185.

[2] P. B. Roemer, W. A. Edelstein, C. E. Hayes, S. P. Souza, O. M. Mueller, The nmr phased array, Magnetic Resonance in Medicine 16 (2) (1990) 192–225.

[3] C. D. Constantinides, E. Atalar, E. R. McVeigh, Signal-to-noise measurements in magnitude images from nmr phased arrays, Magnetic Resonance in Medicine 38 (5) (1997) 852–857.

[4] C. G. Koay, P. J. Basser, Analytically exact correction scheme for signal extraction from noisy magnitude MR signals, Journal of Magnetic Resonance 179 (2) (2006) 317–322.

[5] C. G. Koay, E. Özarslan, C. Pierpaoli, Probabilistic identification and estimation of noise (piesno): A self-consistent approach and its applications in mri, Journal of Magnetic Resonance 199 (1) (2009) 94–103.

[6] J. Sijbers, D. Poot, A. J. d. Dekker, W. Pintjens, Automatic estimation of the noise variance from the histogram of a magnetic resonance image, Physics in Medicine and Biology 52 (5) (2007) 1335–1348.

[7] L.-C. Chang, G. K. Rohde, C. Pierpaoli, An automatic method for estimating noise-induced signal variance in magnitude-reconstructed magnetic resonance images, in: Proc. SPIE, Vol. 5747, SPIE, San Diego, CA, USA, 2005, pp. 1136–1142.